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DYNAMIC ANALYSIS OF A SYSTEM OF HINGE-CONNECTED RIGID BODIES WITH NONRIGID APPENDAGES

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Abstract—Equations of motion are derived for use in simulating a spacecraft or other complex electromechanical system amenable to idealization as a set of hinge-connected rigid bodies of tree topology, with rigid axisymmetric rotors and nonrigid appendages attached to each rigid body in the set. In conjunction with a previously published companion paper on finite-element appendage vibration equations, this paper provides a complete minimum-dimension formulation suitable for generic programming for digital computer numerical integration.

INTRODUCTION

IN A PREVIOUSLY published paper [1] there appear equations of motion which characterize the small, time-varying deformations of an elastic appendage attached to a rigid body experiencing arbitrary motions in inertial space. The flexible appendage is modeled as a set of deformable elastic elements possessing distributed mass and interconnected at N nodes, with a rigid *nodal body* appearing also at each node. This *finite-element* model has $6N$ degrees of freedom in deformation, corresponding to the degrees of freedom invested in the nodal bodies; deformations of the internodal elastic elements are established by assigned interpolation functions. The purpose of [1] is to establish the structure of the $6N$ deformation equations, in order to permit consideration of coordinate transformations which might introduce the possibility of coordinate truncation and the consequent representation of elastic appendage deformations in terms of distributed or modal coordinates numbering much less than $6N$. With this objective accomplished, the referenced paper terminates, leaving a set of equations of motion which are incomplete in the sense that they are insufficient to determine the kinematical variables characterizing the motion in inertial space of the rigid base to which the flexible appendage is attached.

It is the purpose of the present paper not only to complete the dynamic analysis initiated in the earlier work, but to do so in a way that encompasses a wide class of vehicles, namely, those amenable to idealization as a set of $n+1$ rigid bodies interconnected by n line hinges (implying tree topology), with the possibility of rigid axisymmetric rotors and arbitrary nonrigid appendages attached to each rigid body in the set.

The results of the previous paper provide the vibratory deformation equations for each elastic appendage in the system, and the transformation to distributed coordinates which is appropriate in each case. The new equations to be derived in this paper are descriptive of the inertial translations and rotations of one reference body and the relative rotations

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occurring at the n line hinges, with an additional equation being added for each axi-symmetric rotor in the system. In this respect there is a strong parallel between the results of the present paper and those obtained by extending the Hooker–Margulies equations [2] as suggested by Hooker [3] in order to eliminate unwanted *kinematic constraint*† torques; the significant difference is that Hooker and Margulies considered only point-connected *rigid* bodies in a topological tree, whereas in the present paper the basic elements in the tree are substructures which are rigid in part but include rotors and arbitrary nonrigid appendages.

As in [2] and [3], the final equations of motion are presented here in vector-dyadic form. Before they can be committed to a computer program, these equations must be transformed into scalar form or into a matrix form parallel to that generated by Roberson and Wittenburg for the multiple-rigid-body system [4]. A forthcoming Technical Report of the Jet Propulsion Laboratory (JPL) will include matrix equations corresponding to restricted versions of the vector-dyadic equations provided in this paper.

A digital computer program for the numerical integration of the general equations reported here is under development at JPL by G. E. Fleischer. The result is to be a generic program, suitable for the dynamic simulation of a wide class of spacecraft. Many features associated with the following derivation have been adopted so as to minimize the labors of the users of this program.

In comparing the results of the present paper with those which underlie flexible body simulation procedures under development by other organizations, we find the closest counterpart to our work to be that described in [10]. This reference employs an equation assembly algorithm (LEGUP) first introduced some years ago for multi-rigid-body systems by Dr. R. L. Farrenkopf, and this algorithm (applied at each integration step) provides an alternative to the practice of deriving a comprehensive set of generic equations in advance of computation as in the present paper. Presumably, some price in computer run-time is paid for the LEGUP routine. (Dr. R. Gluck of TRW estimated a 10 per cent penalty in a 1971 conversation with this author). Moreover, the referenced equations [10] are expressed in terms of "modal coordinates" defining response in "mode shapes" which are assumed to be real, but otherwise undefined. As shown in [1] and [6], for spinning flexible appendages complex modes are most appropriate. In contrast to [10], in the vector–dyadic equations in the present paper, the "appendage" is restricted only by the assumption of constant mass; it need not be an elastic body, and it is not restricted to small deformations. These equations imply no commitment to modal coordinates, but if such are used they may be real or complex. (Such generality is possible only because this paper stops with vector–dyadic equations and integral representations; the transition to matrix equations and a computer program will necessarily involve further restrictions.)

MATHEMATICAL MODEL

Any problem of dynamic analysis must begin with the adoption of a mathematical model representing the physical system of interest. In what follows, it is assumed that the model consists of $n+1$ rigid bodies (labeled $\ell_0, \ldots, \ell_n$) interconnected by n line hinges

† Kinematic constraint forces and torques are respectively those interaction forces and torques which maintain kinematic constraints, such as "no relative translation of two points" or "no deviation from a prescribed relative rotation."

(implying no closed loops and hence tree topology), with each body containing an arbitrary number (perhaps zero) of rigid rotors, each with an axis of symmetry fixed in the housing body, and moreover with the possibility of attaching to each of the $n+1$ bodies a nonrigid appendage, with appendage $\mathscr{a}_k$ attached to body $\mathscr{b}_k$. The appendage itself can be modeled in a variety of ways without exceeding the scope of the final vector-dyadic equations in this paper; one might adopt a continuum model, a distributed-mass finite-element model, or a model admitting mass only in the form of nodal bodies or nodal particles, and one need not assume small deformations. Specific choices of small deformation appendage models are made in [1], [5] and [6], and equations derived in these references can be used to augment the results of the present paper.

If the actual connection between two massive portions of the physical system admits two (or three) degrees of freedom in rotation, then the analyst simply introduces one (or two) massless and dimensionless imaginary bodies into his model (as though they were massless gimbals). Since the number of equations to be derived here matches the number of degrees of freedom of the system, no price is paid in problem dimension by the introduction of imaginary bodies, and considerable simplification results in the user input format for the computer program.

Each combination of a rigid body and its internal rotors and attached flexible appendage comprises a basic building block referred to here as a *substructure*; thus, there are $n+1$ substructures in the total system, so labeled that $\mathscr{s}_k$ encompasses $\mathscr{b}_k$, $\mathscr{a}_k$, and any rotors in $\mathscr{b}_k$.

DERIVATION PROCEDURE

Step 1

Isolate each substructure and apply

$$\mathbf{F}_T^j = \mathscr{M}_j \mathbf{A}^j \quad \text{and} \quad \mathbf{T}_T^j = \dot{\mathbf{H}}^j \qquad j = 0, 1, \ldots, n \tag{1}$$

where, for the jth substructure, $\mathbf{F}_T^j$ is the total resultant of all external forces, $\mathscr{M}_j$ is the substructure mass, $\mathbf{A}^j$ is the inertial acceleration of the substructure mass center c_j, $\mathbf{T}_T^j$ is the total moment resultant of all external forces referred to c_j, and $\dot{\mathbf{H}}^j$ is the inertial time derivative of the substructure angular momentum referred to c_j.

Step 2

Combine the $6n+6$ scalar equations obtained in Step 1 so as to obtain $n+6$ scalar equations which do not involve redundant variables or those substructure interaction forces or torques which serve to maintain kinematic constraints.

Step 3

Apply $\mathbf{T} = \dot{\mathbf{H}}$ to each rotor of the system (where symbol definitions follow naturally from equation (1)), and dot-multiply each equation by a unit vector parallel to the symmetry axis of the corresponding rotor; the result is a set of scalar equations matching in number the rigid, axisymmetric rotors in the system.

Step 4

Record the appendage deformation equations from Ref. 1 (or an alternative source, depending on the appendage model), and, if appendage deformations are small, substitute the appropriate (truncated) deformation coordinate transformations wherever the deformation variables appear in the preceding equations.

Step 5

Specify all control laws in the form of ordinary differential equations in time, with control torque magnitudes or their equivalent as dependent variables.

Only steps 1–3 are recorded explicitly here, since step 4 is available in [1] and step 5 depends entirely on the specific characteristics of a given vehicle.

DEFINITIONS AND NOTATIONS

Definitions and notations are as follows (see Fig. 1):

(1) Let n be the number of hinges interconnecting a set of $n+1$ substructures.

(2) Define the integer set $\mathscr{B} \triangleq \{0, 1, \ldots, n\}$.

(3) Define the integer set $\mathscr{P} \triangleq \{1, \ldots, n\}$.

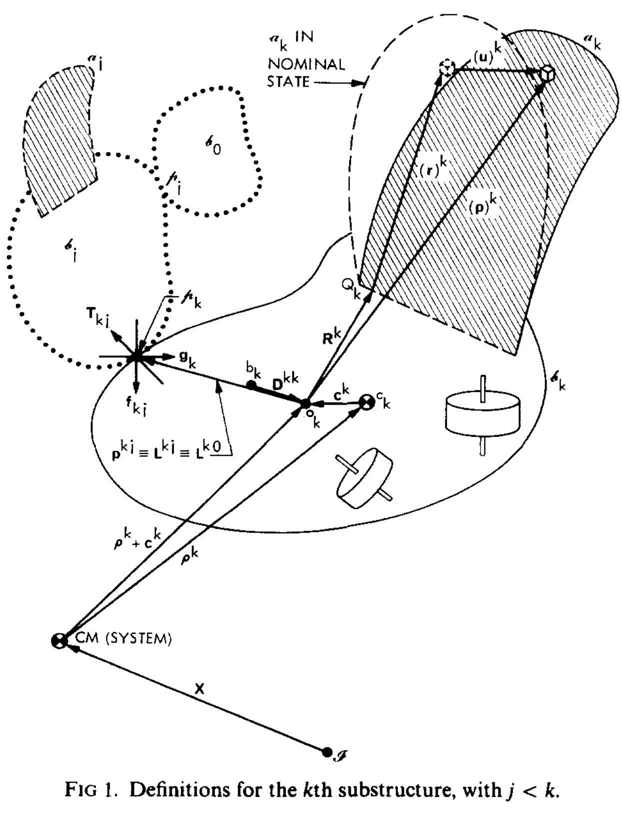

FIG 1. Definitions for the kth substructure, with $j < k$.

(4) Let ℓ_0 be a label assigned to one rigid body chosen arbitrarily as a reference body, and let $\ell_1, \ldots, \ell_n$ be labels assigned to the rest of the rigid bodies in such a way that if ℓ_j is located between ℓ_0 and ℓ_k then $0 < j < k$.
(5) Define dextral, orthogonal sets of unit vectors $\mathbf{b}_1^k, \mathbf{b}_2^k, \mathbf{b}_3^k$ so as to be imbedded in ℓ_k for $k \in \mathscr{B}$, and such that in some arbitrarily selected nominal configuration of the total system $\mathbf{b}_\alpha^k = \mathbf{b}_\alpha^j$ for $\alpha = 1, 2, 3$ and $k, j \in \mathscr{B}$.
(6) Let $\boldsymbol{\omega}^k$ be the inertial angular velocity vector of ℓ_k, for $k \in \mathscr{B}$.
(7) Let c_k be the mass center of the kth substructure, $k \in \mathscr{B}$.
(8) Let $\not{h}_k$ be a point on the hinge axis common to ℓ_k and ℓ_j for $j < k$ and $k \in \mathscr{P}$.
(9) Let $\mathbf{p}^{kj}$ be the position vector of the hinge point connecting ℓ_j and ℓ_k from the point 0_k occupied by c_k when the kth substructure is in its nominal state.
(10) Let $\mathbf{c}^k$ be the position vector from c_k to 0_k.
(11) Let $\boldsymbol{\rho}^k$ be the position vector to c_k from the system mass center CM.
(12) Let $\mathbf{X}$ be the position vector to CM from an inertially fixed point $\mathscr{I}$.
(13) Let $\mathscr{M}_k$ be the mass of the kth substructure, for $k \in \mathscr{B}$.
(14)† Let $(\mathbf{p})^k$ be a generic position vector from 0_k to any point in the kth substructure.
(15)† Let $(\mathbf{u})^k$ be a generic vector of the kth appendage describing the displacement relative to ℓ_k from some nominal state (perhaps an undeformed state).
(16) Let $\mathbf{g}^k$ be a unit vector parallel to the hinge axis through $\not{h}_k$.
(17) For $k \in \mathscr{P}$, let γ_k be the angle of a $\mathbf{g}^k$ rotation of ℓ_k with respect to the body attached at $\not{h}_k$. Let γ_k be zero when $\mathbf{b}_\alpha^k = \mathbf{b}_\alpha^j$ ($\alpha = 1, 2, 3; j, k \in \mathscr{B}$).
(18) Let $\mathbf{J}^k$ be the inertia dyadic of the kth substructure for 0_k, so that $\mathbf{J}^k$ is time variable by virtue of small deformations.
(19) Let $\mathbf{F}^k$ be the resultant vector of all forces applied to the kth substructure except for those due to interbody forces transmitted at hinge connections.
(20) Let $\mathbf{T}^k$ be the resultant moment vector with respect to c_k of all forces applied to the kth substructure except for those due to interbody forces transmitted at hinge connections.
(21) Let τ_k be the scalar magnitude of the torque component applied to ℓ_k in the direction of $\mathbf{g}^k$ by the body attached at $\not{h}_k$.
(22) Let $\mathbf{F} = \sum_{k \in \mathscr{B}} \mathbf{F}^k$ be the external force resultant for the total system.
(23) Define the scalar $\mathscr{E}_{sk}$ such that for $k \in \mathscr{B}$ and $s \in \mathscr{P}$

$$\mathscr{E}_{sk} \triangleq \begin{cases} 1 \text{ if } \not{h}_s \text{ lies between } \ell_0 \text{ and } \ell_k \\ 0 \text{ otherwise} \end{cases}$$

(The $n(n+1)$ scalars $\mathscr{E}_{sk}$ are called *path elements*.)
(24) Define $\mathscr{M} \triangleq \sum_{k \in \mathscr{B}} \mathscr{M}_k$, the total system mass.
(25) Let N_{kr} denote the index of the body attached to ℓ_k and on the path leading to ℓ_r, and let $N_{kk} \triangleq k$. (These are the *network elements*.) For notational simplicity, use N_k for N_{k0}.
(26) For‡ $r \in \mathscr{B} - k$, let $\mathbf{L}^{kr} \triangleq \mathbf{p}^{kN_{kr}}$, and let $\mathbf{L}^{kk} = 0$.
(27) Define $\mathbf{D}^{kk} \triangleq -\sum_{j \in \mathscr{B}} \mathbf{L}^{kj} \mathscr{M}_j / \mathscr{M}$ for $k \in \mathscr{B}$.

† Superscripts on generic symbols such as $\mathbf{p}$ and $\mathbf{u}$ will be omitted when obvious, as when the symbol appears within an integrand of a definite integral.
‡ For notational brevity, the set $\mathscr{B} - \{k\}$ is designated $\mathscr{B} - k$.

(28) Let b_k be a point fixed in ℓ_k such that $\mathbf{D}^{kk}$ is the position vector of 0_k with respect to b_k. (This point b_k is called *barycenter* of the kth substructure in the nominal state.)
(29) Define $\mathbf{D}^{kj} \triangleq \mathbf{D}^{kk} + \mathbf{L}^{kj}$ for $k, j \in \mathscr{B}$.
(30) Define the dyadic $\mathbf{K}^k \triangleq \sum_{r\in\mathscr{B}} \mathscr{M}_r(\mathbf{D}^{kr}\,.\,\mathbf{D}^{kr}\mathbf{U} - \mathbf{D}^{kr}\mathbf{D}^{kr})$ where $\mathbf{U}$ is the unit dyadic.
(31) Define $\mathbf{\Phi}^{kk} \triangleq \mathbf{K}^k + \mathbf{J}^k$.
(32) Define $\mathbf{\Phi}^{kj} \triangleq -\mathscr{M}(\mathbf{D}^{jk}\,.\,\mathbf{D}^{kj}\mathbf{U} - \mathbf{D}^{jk}\mathbf{D}^{kj})$.
(33) Let the system of forces applied to ℓ_k by the attached body ℓ_j be equivalent to a resultant force $\mathbf{f}^{kj}$ passing through the labeled point (p_j or p_k) common to ℓ_k and ℓ_j, plus a torque $\mathbf{T}^{kj}$.
(34) Let $\mathbf{t}^{kj}$ be the kinematic constraint torque applied to ℓ_k by ℓ_j, such that with Definitions 33, 16 and 21,

$$\mathbf{T}^{kj} = \mathbf{t}^{kj} + \delta_{jN_k}\tau_k\mathbf{g}^k - \delta_{kN_j}\tau_j\mathbf{g}^j$$

where

$$\delta_{jj} = 1 \quad \text{and} \quad \delta_{jl} = 0 \quad \text{for } l \neq j$$

(35) Let $\mathscr{B}_r$ be the rth *neighbor set* for $r \in \mathscr{B}$, such that $k \in \mathscr{B}_r$ if ℓ_k is attached to ℓ_r.
(36) Let $\mathscr{B}_{kj}$ be the *branch set* of integers r such that $r \in \mathscr{B}_{kj}$ if $j = N_{kr}$. Thus $\mathscr{B}_{kj}$ consists of the indices of those bodies on a branch which is attached to ℓ_k and begins with ℓ_j.
(37) Let $\mathbf{h}^k$ be the contribution of rotors in ℓ_k to the angular momentum of the kth substructure relative to ℓ_k with respect to 0_k.

DERIVATIONS OF VECTOR–DYADIC EQUATIONS

In terms of the indicated definitions, equation (1) provides for the rth substructure ($r \in \mathscr{B}$):

$$\mathbf{F}^r + \sum_{s\in\mathscr{B}_r} \mathbf{f}^{rs} - \mathscr{M}_r(\ddot{\mathbf{X}} + \ddot{\mathbf{p}}^r) = 0 \tag{2}$$

and for the kth substructure ($k \in \mathscr{B}$):

$$\mathbf{T}^k + \sum_{j\in\mathscr{B}_k} \mathbf{T}^{kj} + \sum_{j\in\mathscr{B}_k} (\mathbf{p}^{kj} + \mathbf{c}^k) \times \mathbf{f}^{kj} - \dot{\mathbf{H}}^k = 0. \tag{3}$$

Here a dot over a vector implies time differentiation in an inertial frame of reference.

As shown in [5, pp. 32–33], the kth substructure angular momentum $\mathbf{H}^k$ can be expanded from basic definitions as

$$\mathbf{H}^k \triangleq \int_{s_k} (\mathbf{c} + \mathbf{p}) \times (\dot{\mathbf{c}} + \dot{\mathbf{p}})\,\mathrm{d}m = \mathbf{J}^k\,.\,\boldsymbol{\omega}^k + \mathscr{M}_k\dot{\mathbf{c}}^k \times \mathbf{c}^k + \int_{s_k} \mathbf{p} \times \mathring{\mathbf{p}}\,\mathrm{d}m \tag{4}$$

where superscripts k are dropped from $\mathbf{c}$ and $\mathbf{p}$ within integrals ranging over the substructure s_k, and where the open circle over a vector implies time differentiation in the reference frame established by ℓ_k. The contribution of a rotor to the integral last appearing in equation (4) may be recognized as a vector fixed in ℓ_k and parallel to the rotor axis, with a magnitude equal to the product of rotor spin axis inertia and the relative angular speed of the rotor relative to ℓ_k; the vector sum of the contributions to this integral of all the rotors in ℓ_k is by Definition 37 designated $\mathbf{h}^k$. Since $\mathring{\mathbf{p}}^k = 0$ for any field point in ℓ_k,

$$\mathbf{H}^k = \mathbf{J}^k \cdot \boldsymbol{\omega}^k + \mathcal{M}_k \dot{\mathbf{c}}^k \times \mathbf{c}^k + \mathbf{h}^k + \int_{a_k} \mathbf{p} \times \mathring{\mathbf{p}} \, \mathrm{d}m \tag{5}$$

limiting the range of integration to the appendage a_k. Substitution of the inertial time derivative of equation (5) into equation (3) produces, for $k \in \mathscr{B}$,

$$\mathbf{T}^k + \sum_{j \in \mathscr{B}_k} \mathbf{T}^{kj} + \sum_{j \in \mathscr{B}_k} \mathbf{p}^{kj} \times \mathbf{f}^{kj} + \mathbf{c}^k \times \sum_{j \in \mathscr{B}_k} \mathbf{f}^{kj} - \frac{^i\mathrm{d}}{\mathrm{d}t}(\mathbf{J}^k \cdot \boldsymbol{\omega}^k)$$
$$- \mathcal{M}_k \ddot{\mathbf{c}}^k \times \mathbf{c}^k - \dot{\mathbf{h}}^k - \frac{^i\mathrm{d}}{\mathrm{d}t} \int_{a_k} \mathbf{p} \times \mathring{\mathbf{p}} \, \mathrm{d}m = 0 \tag{6}$$

where either an overdot or the presuperscript i denotes an inertial reference frame for time differentiation of a vector.

The integral in equation (6) cannot be evaluated explicitly without adopting a specific mathematical model of the flexible appendage; nor can one go beyond the integral representation of the substructure inertia dyadic $\mathbf{J}^k$ and the vector $\mathbf{c}^k$, which vector establishes the shift of the substructure mass center c_k relative to body-fixed point 0_k due to appendage deformation. Without commitment to a particular appendage model, one can accept (dropping subscripts within integrands)

$$\mathbf{J} \triangleq \int_{\mathscr{S}_k} (\mathbf{p} \cdot \mathbf{p} U - \mathbf{p}\mathbf{p}) \, \mathrm{d}m \tag{7}$$

and

$$\mathbf{c}^k \triangleq -\frac{1}{\mathcal{M}_k} \int_{\mathscr{S}_k} \mathbf{u} \, \mathrm{d}m. \tag{8}$$

If the deformations $\mathbf{u}$ are to be assumed "small", the term $-\mathcal{M}_k \ddot{\mathbf{c}}^k \times \mathbf{c}^k$ in equation (6) becomes negligible, and can be ignored.

Since it is possible to manipulate equations (2) and (6) to eliminate redundant variables and kinematic constraint forces and torques without making any restrictive assumptions concerning the appendage model or the size of its deformations, there will be no inhibiting assumptions imposed until specific cases are considered at the conclusion of this section.

The immediate objective is the extraction from the $6n+6$ equations given by equations (2) and (6) of $6+n$ equations in the $6+n$ unknowns established by the vectors $\mathbf{X}$ and $\boldsymbol{\omega}^0$ and the scalars $\gamma_1, \ldots, \gamma_n$ which define the relative rotations of contiguous rigid bodies. We must expect these equations also to involve unknown deformation variables for the appendages and rate variables for the rotors, but we must eliminate all interaction forces (typified by $\mathbf{f}^{kj}$) and all interaction torques (typified by $\mathbf{T}^{kj}$) *except* for those having the direction of the corresponding hinge axes, since these hinge-axis torques are known functions of hinge rotation or some more general control law. We must also eliminate the unknown position vectors typified by $\boldsymbol{\rho}^r$, replacing them by explicit functions of the hinge angles $\gamma_1, \ldots, \gamma_n$ and the system geometry, and we must eliminate all $\boldsymbol{\omega}^k$ for $k \in \mathscr{P}$ in favor of terms involving $\boldsymbol{\omega}^0$ and $\dot{\gamma}_r$ for $r \in \mathscr{P}$.

Simply by summing all equations defined by equations (2), we obtain the familiar result

$$\mathbf{F} = \mathcal{M} \ddot{\mathbf{X}} \tag{9}$$

where $\mathscr{M}$ is the total system mass and $\mathbf{F}$ is the resultant of all external forces $\mathbf{F}^r$ for $r \in \mathscr{B}$. Substituting from equation (9) back into equation (2) and rearranging produces, for $r \in \mathscr{B}$,

$$\sum_{s \in \mathscr{B}_r} \mathbf{f}^{rs} = -\mathbf{F}^r + \mathscr{M}_r(\mathbf{F}/\mathscr{M} + \ddot{\boldsymbol{\rho}}^r). \tag{10}$$

Summing over the branch set $\mathscr{B}_{kj}$ (see Definition 36) provides

$$\mathbf{f}^{jk} = \sum_{r \in \mathscr{B}_{kj}} \sum_{s \in \mathscr{B}_r} \mathbf{f}^{rs} = -\sum_{r \in \mathscr{B}_{kj}} [\mathbf{F}^r - \mathscr{M}_r(\mathbf{F}/\mathscr{M} + \ddot{\boldsymbol{\rho}}^r)]$$

so that, by Newton's third law,

$$\mathbf{f}^{kj} = -\mathbf{f}^{jk} = \sum_{r \in \mathscr{B}_{kj}} [\mathbf{F}^r - \mathscr{M}_r(\mathbf{F}/\mathscr{M} + \ddot{\boldsymbol{\rho}}^r)]. \tag{11}$$

Substituting equations (10) and (11) into equation (6) eliminates all interaction forces from the rotational equations, providing

$$\mathbf{T}^k + \sum_{j \in \mathscr{B}_k} \mathbf{T}^{kj} + \sum_{j \in \mathscr{B}_k} \mathbf{p}^{kj} \times \sum_{r \in \mathscr{B}_{kj}} [\mathbf{F}^r - \mathscr{M}_r(\mathbf{F}/\mathscr{M} + \ddot{\boldsymbol{\rho}}^r)] + \mathbf{c}^k \times [-\mathbf{F}^k + \mathscr{M}_k(\mathbf{F}/\mathscr{M} + \ddot{\boldsymbol{\rho}}^k)]$$

$$-\frac{{}^i d}{dt}(\mathbf{J}^k \cdot \boldsymbol{\omega}^k) - \mathscr{M}_k \ddot{\mathbf{c}}^k \times \mathbf{c}^k - \dot{\mathbf{h}}^k - \frac{{}^i d}{dt} \int_{a_k} \mathbf{p} \times \mathring{\mathbf{p}}\, dm = 0 \qquad (k \in \mathscr{B}). \tag{12}$$

Equation (12) simplifies when written in terms of the vectors found in Definitions 26–29 in the preceding section, due to the identity, for $k \in \mathscr{B}$

$$\begin{aligned}\sum_{j \in \mathscr{B}_k} \mathbf{p}^{kj} \times \sum_{r \in \mathscr{B}_{kj}} [\mathbf{F}^r - \mathscr{M}_r(\mathbf{F}/\mathscr{M} + \ddot{\boldsymbol{\rho}}^r)] &= \sum_{r \in \mathscr{B}} \mathbf{L}^{kr} \times [\mathbf{F}^r - \mathscr{M}_r(\mathbf{F}/\mathscr{M} + \ddot{\boldsymbol{\rho}}^r)] \\ &= \sum_{r \in \mathscr{B}} \mathbf{L}^{kr} \times \mathbf{F}^r + \mathbf{D}^{kk} \times \mathbf{F} - \sum_{r \in \mathscr{B}} \mathbf{L}^{kr} \times \mathscr{M}_r \ddot{\boldsymbol{\rho}}^r \\ &= \sum_{r \in \mathscr{B}} (\mathbf{L}^{kr} + \mathbf{D}^{kk}) \times \mathbf{F}^r - \sum_{r \in \mathscr{B}} \mathbf{L}^{kr} \times \mathscr{M}_r \ddot{\boldsymbol{\rho}}^r \\ &= \sum_{r \in \mathscr{B}} (\mathbf{D}^{kr} \times \mathbf{F}^r - \mathbf{L}^{kr} \times \mathscr{M}_r \ddot{\boldsymbol{\rho}}^r). \end{aligned} \tag{13}$$

Substituting equation (13) into equation (12) simplifies its appearance somewhat, but there remains the problem of eliminating the unknown kinematic constraint torques which are present within the interaction torques typified by $\mathbf{T}^{kj}$. By summing over all $n+1$ vector equations defined by equation (12), we can by virtue of Newton's third law obtain one vector equation involving no interaction torques at all; this summation gives

$$\sum_{k \in \mathscr{B}} \Big\{ \mathbf{T}^k + \sum_{r \in \mathscr{B}} (\mathbf{D}^{kr} \times \mathbf{F}^r - \mathbf{L}^{kr} \times \mathscr{M}_r \ddot{\boldsymbol{\rho}}^r) + \mathbf{c}_k \times [\mathscr{M}_k(\ddot{\boldsymbol{\rho}}^k + \mathbf{F}/\mathscr{M}) - \mathbf{F}^k]$$

$$-\frac{{}^i d}{dt}(\mathbf{J}^k \cdot \boldsymbol{\omega}^k) - \mathscr{M}_k \ddot{\mathbf{c}}^k \times \mathbf{c}^k - \dot{\mathbf{h}}^k - \frac{{}^i d}{dt} \int_{a_k} \mathbf{p} \times \mathring{\mathbf{p}}\, dm \Big\} = 0. \tag{14}$$

Equation (14), like equation (9), is free of kinematic constraint forces and torques, so that these two vector equations can be preserved in the final set that is our objective. The remaining n scalar equations can be obtained by summing equations obtained from equation (12) over branch sets in order to isolate interaction torques such as $\mathbf{T}^{kj}$, and then by dot-multiplying each expression by the unit vector parallel to the corresponding hinge axis. In particular, if we introduce the network elements found in Definition 25 of the previous section, we can sum over the branch set $\mathscr{B}_{N_s s}$ and get, for $s \in \mathscr{B}$,

$$\mathbf{T}^{sN_s}+\sum_{k\in\mathscr{B}_{N_s s}}\Big\{\mathbf{T}^k+\sum_{r\in\mathscr{B}}(\mathbf{D}^{kr}\times\mathbf{F}^r-\mathbf{L}^{kr}\times\mathscr{M}_r\ddot{\boldsymbol{\rho}}^r)+\mathbf{c}^k\times[\mathscr{M}_k(\ddot{\boldsymbol{\rho}}^k+\mathbf{F}/\mathscr{M})-\mathbf{F}^k]$$
$$-\frac{{}^i d}{dt}(\mathbf{J}^k\cdot\boldsymbol{\omega}^k)-\mathscr{M}_k\ddot{\mathbf{c}}^k\times\mathbf{c}^k-\dot{\mathbf{h}}^k-\frac{{}^i d}{dt}\int_{a_k}\mathbf{p}\times\mathring{\mathbf{p}}\,dm\Big\}=0. \tag{15}$$

By Definition 25, $N_s < s$. Thus the labeled point on the hinge axis common to b_s and b_{N_s} is, by Definition 8, called p_s; the unit vector parallel to the hinge axis is, by Definition 16, called $\mathbf{g}^s$; and the interaction torque component of $\mathbf{T}^{sN_s}$ parallel to $\mathbf{g}^s$ is, by Definition 21, called τ_s. Hence the dot product of equation (15) with $\mathbf{g}^s$ provides the scalar equation, for $s \in \mathscr{B}$,

$$\tau_s+\mathbf{g}^s\cdot\sum_{k\in\mathscr{B}_{N_s s}}\Big\{\mathbf{T}^k+\sum_{r\in\mathscr{B}}(\mathbf{D}^{kr}\times\mathbf{F}^r-\mathbf{L}^{kr}\times\mathscr{M}_r\ddot{\boldsymbol{\rho}}^r)+\mathbf{c}^k\times[\mathscr{M}_k(\ddot{\boldsymbol{\rho}}^k+\mathbf{F}/\mathscr{M})-\mathbf{F}^k]$$
$$-\frac{{}^i d}{dt}(\mathbf{J}^k\cdot\boldsymbol{\omega}^k)-\mathscr{M}_k\ddot{\mathbf{c}}^k\times\mathbf{c}^k-\dot{\mathbf{h}}^k-\frac{{}^i d}{dt}\int_{a_k}\mathbf{p}\times\mathring{\mathbf{p}}\,dm\Big\}=0. \tag{16}$$

Equations (9), (14) and (16) provide the required $6+n$ scalar equations free of kinematic constraint forces and torques, but they are not yet in final vector–dyadic form. The variable vectors typified by $\boldsymbol{\rho}^r$ must first be expressed in terms of the system geometry, the deformation variables, and a subset of the $6+n$ scalars required to define $\mathbf{X}$, $\boldsymbol{\omega}^0$, and the n angles of relative rotation at the n hinges. As a first step in this direction, we define $\mathscr{C}_{rj}$ as the set of indices of bodies lying on the direct path between b_r and b_j, and then (referring to Fig. 2) we can write

$$\boldsymbol{\rho}^r-\boldsymbol{\rho}^j = \mathbf{c}^j+\mathbf{L}^{jr}+\sum_{s\in\mathscr{C}_{rj}}(\mathbf{L}^{sr}-\mathbf{L}^{sj})-\mathbf{L}^{rj}-\mathbf{c}^r. \tag{17}$$

But $\mathbf{L}^{jj} = \mathbf{L}^{rr} = 0$, by Definition 26, and for any index s in the set $\mathscr{B}-\mathscr{C}_{rj}$ the sum $\mathbf{L}^{sr}-\mathbf{L}^{sj} = 0$, so that equation (18) can be written more simply as

$$\boldsymbol{\rho}^r-\boldsymbol{\rho}^j = \sum_{s\in\mathscr{B}}(\mathbf{L}^{sr}-\mathbf{L}^{sj})+\mathbf{c}^j-\mathbf{c}^r. \tag{18}$$

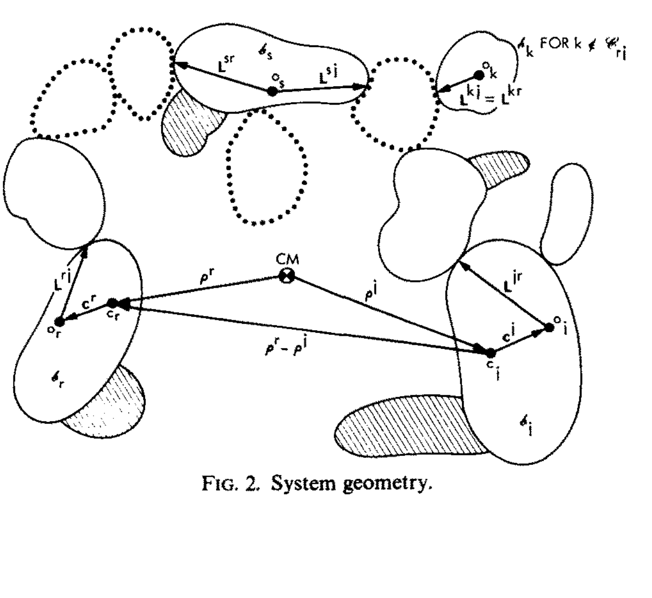

FIG. 2. System geometry.

Multiplying equation (18) by $\mathcal{M}_j/\mathcal{M}$ and summing over all $j \in \mathcal{B}$ produces

$$\sum_{j\in\mathcal{B}} \frac{\mathcal{M}_j}{\mathcal{M}}\boldsymbol{\rho}^r - \sum_{j\in\mathcal{B}}\frac{\mathcal{M}_j}{\mathcal{M}}\boldsymbol{\rho}^j = \sum_{j\in\mathcal{B}}\sum_{s\in\mathcal{B}}\frac{\mathcal{M}_j}{\mathcal{M}}(\mathbf{L}^{sr}-\mathbf{L}^{sj}) + \sum_{j\in\mathcal{B}}\frac{\mathcal{M}_j}{\mathcal{M}}(\mathbf{c}^j-\mathbf{c}^r).$$

But $\sum_{j\in\mathcal{B}} \mathcal{M}_j\boldsymbol{\rho}^j = 0$ by definition of CM, and $\sum_{j\in\mathcal{B}}\mathcal{M}_j \triangleq \mathcal{M}$, so we have

$$\boldsymbol{\rho}^r = \sum_{j\in\mathcal{B}}\sum_{s\in\mathcal{B}}\frac{\mathcal{M}_j}{\mathcal{M}}(\mathbf{L}^{sr}-\mathbf{L}^{sj}) + \frac{1}{\mathcal{M}}\sum_{j\in\mathcal{B}}\mathcal{M}_j\mathbf{c}^j - \mathbf{c}^r.$$

Reversing the summation sequence in the first term, changing the index symbol in the final summation, and substituting the vectors introduced in Definitions 27 and 29, we find simply that, for $r \in \mathcal{B}$,

$$\boldsymbol{\rho}^r = \sum_{s\in\mathcal{B}}\left(\mathbf{L}^{sr}+\mathbf{D}^{ss}+\frac{\mathcal{M}_s}{\mathcal{M}}\mathbf{c}^s\right) - \mathbf{c}^r = \sum_{s\in\mathcal{B}}\left(\mathbf{D}^{sr}+\frac{\mathcal{M}_s}{\mathcal{M}}\mathbf{c}^s\right) - \mathbf{c}^r. \tag{19}$$

Equations (14) and (16) contain not $\boldsymbol{\rho}^r$ itself but combinations which, with equation (19), can be written as

$$\mathbf{c}^k \times \mathcal{M}_k\ddot{\boldsymbol{\rho}}^k = \mathcal{M}_k\mathbf{c}^k \times \left[\sum_{r\in\mathcal{B}}\left(\ddot{\mathbf{D}}^{rk}+\frac{\mathcal{M}_r}{\mathcal{M}}\ddot{\mathbf{c}}^r\right) - \ddot{\mathbf{c}}^k\right] \tag{20}$$

and, in terms employing the symbols of Definitions 26–30,

$$\begin{aligned}
-\sum_{r\in\mathcal{B}}\mathbf{L}^{kr}\times\mathcal{M}_r\ddot{\boldsymbol{\rho}}^r &= -\sum_{r\in\mathcal{B}}\mathbf{L}^{kr}\times\mathcal{M}_r\left[\sum_{s\in\mathcal{B}}\left(\ddot{\mathbf{D}}^{sr}+\frac{\mathcal{M}_s}{\mathcal{M}}\ddot{\mathbf{c}}^s\right)-\ddot{\mathbf{c}}^r\right]\\
&= -\sum_{r\in\mathcal{B}}\mathbf{L}^{kr}\times\mathcal{M}_r\left[\ddot{\mathbf{D}}^{kr}+\sum_{s\in\mathcal{B}-k}\ddot{\mathbf{D}}^{sr}+\sum_{s\in\mathcal{B}}\frac{\mathcal{M}_s}{\mathcal{M}}\ddot{\mathbf{c}}^s-\ddot{\mathbf{c}}^r\right]\\
&= \sum_{r\in\mathcal{B}}(\mathbf{D}^{kk}-\mathbf{D}^{kr})\times\left[\mathcal{M}_r\ddot{\mathbf{D}}^{kr}+\frac{\mathcal{M}_r}{\mathcal{M}}\left(\sum_{s\in\mathcal{B}}\mathcal{M}_s\ddot{\mathbf{c}}^s-\mathcal{M}\ddot{\mathbf{c}}^r\right)\right]\\
&\quad -\sum_{r\in\mathcal{B}-k}\sum_{s\in\mathcal{B}-k-r}\mathcal{M}_r\mathbf{L}^{kr}\times\ddot{\mathbf{D}}^{sr}-\sum_{r\in\mathcal{B}-k}\mathcal{M}_r\mathbf{L}^{kr}\times\ddot{\mathbf{D}}^{rr}\\
&= \mathbf{D}^{kk}\times\sum_{r\in\mathcal{B}}\mathcal{M}_r\ddot{\mathbf{D}}^{kr}-\frac{{}^{i}d}{dt}\left(\sum_{r\in\mathcal{B}}\mathbf{D}^{kr}\times\mathcal{M}_r\dot{\mathbf{D}}^{kr}\right)+\mathbf{D}^{kk}\times\sum_{s\in\mathcal{B}}\mathcal{M}_s\ddot{\mathbf{c}}^s\\
&\quad -\mathbf{D}^{kk}\times\sum_{r\in\mathcal{B}}\mathcal{M}_r\ddot{\mathbf{c}}^r-\left(\sum_{r\in\mathcal{B}}\mathbf{D}^{kr}\mathcal{M}_r\right)\sum_{s\in\mathcal{B}}\frac{\mathcal{M}_s}{\mathcal{M}}\ddot{\mathbf{c}}^s\\
&\quad +\sum_{r\in\mathcal{B}}\mathbf{D}^{kr}\times\mathcal{M}_r\ddot{\mathbf{c}}^r-\sum_{r\in\mathcal{B}-k}\sum_{s\in\mathcal{B}-k-r}\mathcal{M}_r\mathbf{L}^{kr}\times\ddot{\mathbf{D}}^{sr}+\sum_{r\in\mathcal{B}-k}\sum_{s\in\mathcal{B}-r}\mathcal{M}_s\mathbf{L}^{kr}\times\ddot{\mathbf{D}}^{rs}\\
&= 0-\frac{{}^{i}d}{dt}\left[\sum_{r\in\mathcal{B}}\mathcal{M}_r\mathbf{D}^{kr}\times(\boldsymbol{\omega}^k\times\mathbf{D}^{kr})\right]+0-0+\sum_{r\in\mathcal{B}}\mathbf{D}^{kr}\times\mathcal{M}_r\ddot{\mathbf{c}}^r\\
&\quad -\sum_{r\in\mathcal{B}-k}\sum_{s\in\mathcal{B}-k-r}\mathcal{M}_s\mathbf{L}^{ks}\times\ddot{\mathbf{D}}^{rs}+\sum_{r\in\mathcal{B}-k}\sum_{s\in\mathcal{B}-r}\mathcal{M}_s\mathbf{L}^{kr}\times\ddot{\mathbf{D}}^{rs}\\
&= -\frac{{}^{i}d}{dt}\left[\left\{\sum_{r\in\mathcal{B}}\mathcal{M}_r(\mathbf{D}^{kr}\cdot\mathbf{D}^{kr}\mathbf{U}-\mathbf{D}^{kr}\mathbf{D}^{kr})\right\}\cdot\boldsymbol{\omega}^k\right]\\
&\quad +\sum_{r\in\mathcal{B}}\mathcal{M}_r\mathbf{D}^{kr}\times\ddot{\mathbf{c}}^r+\sum_{r\in\mathcal{B}-k}\sum_{s\in\mathcal{B}-r}\mathcal{M}_s(\mathbf{L}^{kr}-\mathbf{L}^{ks})\times\ddot{\mathbf{D}}^{rs}
\end{aligned}$$

or

$$-\sum_{r\in\mathscr{B}} \mathbf{L}^{kr}\times \mathscr{M}_r\ddot{\boldsymbol{\rho}}^r = -\frac{{}^i d}{dt}[\mathbf{K}^k\cdot\boldsymbol{\omega}^k]+\sum_{r\in\mathscr{B}}\mathscr{M}_r\mathbf{D}^{kr}\times\ddot{\mathbf{c}}^r+\sum_{r\in\mathscr{B}-k}\sum_{s\in\mathscr{B}}\mathscr{M}_s(\mathbf{L}^{kr}-\mathbf{L}^{ks})\times\ddot{\mathbf{D}}^{rs} \quad (21)$$

In the development leading to equation (21), it is recognized that $\sum_{s\in\mathscr{B}}\mathscr{M}_s\mathbf{D}^{rs}=0$ (reflecting the significance of the barycenter as the mass center of the undeformed augmented body, which consists of the substructure augmented at each connection point by a particle having the mass of the corresponding branch set.) It was also noted that

$$-\sum_{r\in\mathscr{B}-k}\sum_{s\in\mathscr{B}-k-r}\mathscr{M}_r\mathbf{L}^{kr}\times\ddot{\mathbf{D}}^{sr} = -\sum_{s\in\mathscr{B}-k}\sum_{r\in\mathscr{B}-k-s}\mathscr{M}_r\mathbf{L}^{kr}\times\ddot{\mathbf{D}}^{sr}$$
$$= -\sum_{r\in\mathscr{B}-k}\sum_{s\in\mathscr{B}-k-r}\mathscr{M}_r\mathbf{L}^{ks}\times\ddot{\mathbf{D}}^{rs}$$

where the second step involves simply relabeling indices.

Finally, we can recognize in equation (21) that the quantity $\mathbf{L}^{kr}-\mathbf{L}^{ks}$ is zero for any index s corresponding to a body which lies anywhere on the branch which begins with $\mathscr{b}_{N_{kr}}$ and includes $\mathscr{b}_r$, and for any other index the quantity $\mathbf{D}^{rs}$ is also $\mathbf{D}^{rk}$. Thus in equation (21), $\ddot{\mathbf{D}}^{rs}$ can be replaced by $\ddot{\mathbf{D}}^{rk}$, with the consequence

$$-\sum_{r\in\mathscr{B}}\mathbf{L}^{kr}\times\mathscr{M}_r\ddot{\boldsymbol{\rho}}^r = -\frac{{}^i d}{dt}(\mathbf{K}^k\cdot\boldsymbol{\omega}^k)+\sum_{r\in\mathscr{B}}\mathscr{M}_r\mathbf{D}^{kr}\times\ddot{\mathbf{c}}^r+\sum_{r\in\mathscr{B}-k}\sum_{s\in\mathscr{B}}\mathscr{M}_s(\mathbf{L}^{kr}-\mathbf{L}^{ks})\times\ddot{\mathbf{D}}^{rk}$$
$$= -\frac{{}^i d}{dt}(\mathbf{K}^k\cdot\boldsymbol{\omega}^k)+\sum_{r\in\mathscr{B}}\mathscr{M}_r\mathbf{D}^{kr}\times\ddot{\mathbf{c}}^r+\sum_{r\in\mathscr{B}-k}\left[\left(\mathscr{M}\mathbf{L}^{kr}-\sum_{s\in\mathscr{B}}\mathscr{M}_s\mathbf{L}^{ks}\right)\times\ddot{\mathbf{D}}^{rk}\right]$$

or, with Definitions 27 and 29,

$$-\sum_{r\in\mathscr{B}}\mathbf{L}^{kr}\times\mathscr{M}_r\ddot{\boldsymbol{\rho}}^r = -\frac{{}^i d}{dt}(\mathbf{K}^k\cdot\boldsymbol{\omega}^k)+\sum_{r\in\mathscr{B}}\mathscr{M}_r\mathbf{D}^{kr}\times\ddot{\mathbf{c}}^r+\sum_{r\in\mathscr{B}-k}\mathscr{M}\mathbf{D}^{kr}\times\ddot{\mathbf{D}}^{rk}. \quad (22)$$

By substituting equations (20) and (22) into equations (14) and (16), we can obtain the desired vector–dyadic equations. In this substitution, it becomes apparent that $\mathbf{K}^k$ and $\mathbf{J}^k$ always appear in combination, suggesting the introduction of the new dyadic $\boldsymbol{\Phi}^{kk}$ (see Definition 31). Then the new versions of equations (14) and (16) become

$$\sum_{k\in\mathscr{B}}\left\{\mathbf{T}^k+\sum_{r\in\mathscr{B}}\mathbf{D}^{kr}\times\mathbf{F}^r+\mathbf{c}^k\times\left(\frac{\mathscr{M}_k}{\mathscr{M}}\mathbf{F}-\mathbf{F}^k\right)+\sum_{r\in\mathscr{B}}\mathscr{M}_r\mathbf{D}^{kr}\times\ddot{\mathbf{c}}^r\right.$$
$$\left.+\mathscr{M}_k\mathbf{c}^k\times\sum_{r\in\mathscr{B}}\left(\ddot{\mathbf{D}}^{rk}+\frac{\mathscr{M}_r}{\mathscr{M}}\ddot{\mathbf{c}}^r\right)-\frac{{}^i d}{dt}(\boldsymbol{\Phi}^{kk}\cdot\boldsymbol{\omega}^k)+\sum_{r\in\mathscr{B}-k}\mathscr{M}\mathbf{D}^{kr}\times\ddot{\mathbf{D}}^{rk}-\dot{\mathbf{h}}^k-\frac{{}^i d}{dt}\int_{a_k}\mathbf{p}\times\mathring{\mathbf{p}}\,dm\right\}=0 \quad (23)$$

and, with judicious use of the path elements found in Definition 23,

$$\tau_s+\mathbf{g}_s\cdot\sum_{k\in\mathscr{P}}\mathscr{E}_{sk}\left\{\mathbf{T}^k+\sum_{r\in\mathscr{B}}\mathbf{D}^{kr}\times\mathbf{F}^r+\mathbf{c}^k\times\left(\frac{\mathscr{M}_k}{\mathscr{M}}\mathbf{F}-\mathbf{F}^k\right)\right.$$
$$+\sum_{r\in\mathscr{B}}\mathscr{M}_r\mathbf{D}^{kr}\times\ddot{\mathbf{c}}^r+\mathscr{M}_k\mathbf{c}^k\times\sum_{r\in\mathscr{B}}\left(\ddot{\mathbf{D}}^{rk}+\frac{\mathscr{M}_r}{\mathscr{M}}\ddot{\mathbf{c}}^r\right)-\frac{{}^i d}{dt}(\boldsymbol{\Phi}^{kk}\cdot\boldsymbol{\omega}^k)$$
$$\left.+\sum_{r\in\mathscr{B}-k}\mathscr{M}\mathbf{D}^{kr}\times\ddot{\mathbf{D}}^{rk}-\dot{\mathbf{h}}^k-\frac{{}^i d}{dt}\int_{a_k}\mathbf{p}\times\mathring{\mathbf{p}}\,dm\right\}=0. \quad (24)$$

Equations (23) and (24) can be restated in more useful form by expanding terms involving time derivatives relative to inertial space to obtain time derivatives relative to reference frames established by individual substructures, plus additional undifferentiated terms.

In particular, noting that $\mathbf{D}^{rk}$ is fixed in ℓ_r, we can substitute

$$\ddot{\mathbf{D}}^{rk} = \dot{\boldsymbol{\omega}}^r \times \mathbf{D}^{rk} + \boldsymbol{\omega}^r \times (\boldsymbol{\omega}^r \times \mathbf{D}^{rk}) \tag{25}$$

so that in terms of Definition 32, we have

$$\sum_{r\in\mathscr{B}-k} \mathscr{M}\mathbf{D}^{kr} \times \ddot{\mathbf{D}}^{rk} = -\sum_{r\in\mathscr{B}-k} \Phi^{kr} \,.\, \dot{\boldsymbol{\omega}}^r + \mathscr{M} \sum_{r\in\mathscr{B}-k} \mathbf{D}^{kr} \times [\boldsymbol{\omega}^r \times (\boldsymbol{\omega}^r \times \mathbf{D}^{rk})]. \tag{26}$$

The parallel expression for $\ddot{\mathbf{c}}^r$ becomes

$$\ddot{\mathbf{c}}^r = \mathring{\mathring{\mathbf{c}}}^r + 2\boldsymbol{\omega}^r \times \mathring{\mathbf{c}}^r + \dot{\boldsymbol{\omega}}^r \times \mathbf{c}^r + \boldsymbol{\omega}^r \times (\boldsymbol{\omega}^r \times \mathbf{c}^r) \tag{27}$$

with open circles representing time differentiation in the reference frame established by ℓ_r.

Other time differentiations in inertial space expand as

$$\dot{\mathbf{h}}^k = \mathring{\mathbf{h}}^k + \boldsymbol{\omega}^k \times \mathbf{h}^k \tag{28}$$

$$\frac{{}^i d}{dt}(\Phi^{kk} \,.\, \boldsymbol{\omega}^k) = \Phi^{kk} \,.\, \dot{\boldsymbol{\omega}}^k + \boldsymbol{\omega}^k \times \Phi^{kk} \,.\, \boldsymbol{\omega}^k + \mathring{\Phi}^{kk} \,.\, \boldsymbol{\omega}^k \tag{29}$$

and

$$\frac{{}^i d}{dt} \int_{a_k} \mathbf{p} \times \mathring{\mathbf{p}} \, dm = \int_{a_k} \mathbf{p} \times \mathring{\mathring{\mathbf{p}}} \, dm + \boldsymbol{\omega}^k \times \int_{a_k} (\mathbf{p} \times \mathring{\mathbf{p}}) \, dm \tag{30}$$

with the open circle indicating time differentiation in the reference frame established by the local substructure (here ℓ_k).

By combining equations (23)–(30), we can obtain the vector dyadic equations of rotation in the form

$$\sum_{k\in\mathscr{B}} \mathbf{W}^k = 0 \tag{31}$$

and

$$\tau_s + \mathbf{g}^s \,.\, \sum_{k\in\mathscr{P}} \mathscr{E}_{sk} \mathbf{W}^k = 0 \tag{32}$$

where

$$\begin{aligned}
\mathbf{W}^k \triangleq\; & \mathbf{T}^k + \sum_{r\in\mathscr{B}} \mathbf{D}^{kr} \times \mathbf{F}^r + \mathbf{c}^k \times \left(\frac{\mathscr{M}_k}{\mathscr{M}} \mathbf{F} - \mathbf{F}^k \right) \\
& + \sum_{r\in\mathscr{B}} \mathscr{M}_r \mathbf{D}^{kr} \times [\mathring{\mathring{\mathbf{c}}}^r + 2\boldsymbol{\omega}^r \times \mathring{\mathbf{c}}^r + \dot{\boldsymbol{\omega}}^r \times \mathbf{c}^r + \boldsymbol{\omega}^r \times (\boldsymbol{\omega}^r \times \mathbf{c}^r)] \\
& + \mathscr{M}_k \mathbf{c}^k \times \sum_{r\in\mathscr{B}} [\dot{\boldsymbol{\omega}}^r \times \mathbf{D}^{rk} + \boldsymbol{\omega}^r \times (\boldsymbol{\omega}^r \times \mathbf{D}^{rk})] \\
& + \mathscr{M}_k \mathbf{c}^k \times \sum_{r\in\mathscr{B}} \frac{\mathscr{M}_r}{\mathscr{M}} [\mathring{\mathring{\mathbf{c}}}^r + 2\boldsymbol{\omega}^r \times \mathring{\mathbf{c}}^r + \dot{\boldsymbol{\omega}}^r \times \mathbf{c}^r + \boldsymbol{\omega}^r \times (\boldsymbol{\omega}^r \times \mathbf{c}^r)] - \Phi^{kk} \,.\, \dot{\boldsymbol{\omega}}^k \\
& - \sum_{r\in\mathscr{B}-k} (\Phi^{kr} \,.\, \dot{\boldsymbol{\omega}}^r) + \mathscr{M} \sum_{r\in\mathscr{B}-k} \mathbf{D}^{kr} \times [\boldsymbol{\omega}^r \times (\boldsymbol{\omega}^r \times \mathbf{D}^{rk})] - \boldsymbol{\omega}^k \times \Phi^{kk} \,.\, \boldsymbol{\omega}^k \\
& - \mathring{\mathbf{h}}^k - \boldsymbol{\omega}^k \times \mathbf{h}^k - \mathring{\Phi}^{kk} \,.\, \boldsymbol{\omega}^k - \int_{a_k} \mathbf{p} \times \mathring{\mathring{\mathbf{p}}} \, dm - \boldsymbol{\omega}^k \times \int_{a_k} (\mathbf{p} \times \mathring{\mathbf{p}}) \, dm.
\end{aligned} \tag{33}$$

To obtain the vector–dyadic equations in their final form, involving the $6+n$ unknowns in $\mathbf{X}$, $\boldsymbol{\omega}^0$, and $\gamma_1, \ldots, \gamma_n$ as well as the rotor momentum variables and the deformation variables, we must substitute in equation (33) the expressions

$$\boldsymbol{\omega}^k = \boldsymbol{\omega}^0 + \sum_{r \in \mathscr{P}} \mathscr{E}_{rk} \dot{\gamma}_r \mathbf{g}^r \tag{34}$$

and

$$\dot{\boldsymbol{\omega}}^k = \dot{\boldsymbol{\omega}}^0 + \sum_{r \in \mathscr{P}} \mathscr{E}_{rk} \left[\ddot{\gamma}_r \mathbf{g}^r + \boldsymbol{\omega}^0 \times \dot{\gamma}_r \mathbf{g}^r + \sum_{s \in \mathscr{P}} \mathscr{E}_{sr} \dot{\gamma}_s \mathbf{g}^s \times \mathbf{g}^r \right] \tag{35}$$

where $\mathscr{E}_{rk}$ are the path elements found in Definition 23, and Definitions 6, 16 and 17 are also employed.

In combination, equations (9) and (31)–(35) provide the final form of the vector–dyadic equations of vehicle translation and substructure rotation. In addition to these equations, for each rotor in the system an additional scalar unknown is introduced, and one more differential equation is required. If a rigid axisymmetric rotor with spin axis inertia $\mathscr{J}_{ks}$ spins relative to body ℓ_k at the angular rate $\dot{\psi}_{ks}$ about an axis parallel to the unit vector $\mathbf{b}^{ks}$ fixed in ℓ_k, and if τ_{ks}^R is the magnitude of the $\mathbf{b}^{ks}$ component of the torque applied to the rotor, then the required additional equation of motion is

$$\tau_{ks}^R = \mathscr{J}_{ks}(\ddot{\psi}_{ks} + \mathbf{b}^{ks} \cdot \dot{\boldsymbol{\omega}}^k). \tag{36}$$

If there are N^k rotors in ℓ_k, and N^R rotors in the entire system, then equation (36) applies for $s = 1, \ldots, N^k$, with $k \in \mathscr{R}$, so that equation (36) contributes N^R scalar equations. Moreover, the variables in equations typified by equation (36) are related to those in equation (33) by

$$\mathbf{h}^k = \sum_{s=1}^{N^k} \mathscr{J}_{ks} \dot{\psi}_{ks} \mathbf{b}^{ks}. \tag{37}$$

The scalars typified by τ_{ks}^R must of course be given, either explicitly or implicitly in the form of additional differential equations representing control laws. The same is true of τ_s, $\mathbf{T}^k$ and $\mathbf{F}^r$ in equations (33) and (32). Once these issues are settled, however, there remains the problem of defining the nonrigid appendages mathematically and establishing their equations of deformational motion. The relative advantages of alternative appendage models are considered in detail in [6], and briefly summarized here.

If the appendage a_k is of extremely simple configuration (such as a uniform elastic beam), and the nominal angular velocity $\boldsymbol{\omega}_k$ of the base to which it is attached is of a special class (such as zero, or parallel or orthogonal to the hypothesized beam), then it might become attractive to model the appendage as an elastic continuum, writing partial differential equations of vibration, seeking normal modes of vibration, and truncating to a modest number of coordinates and a corresponding number of ordinary scalar differential equations. One must then return to those terms in equations (31)–(35) which depend on deformation (noting equations (7) and (8)), and evaluate those terms as functions of the modal deformation coordinates, ignoring second degree terms in deformation variables.

In most cases, it will prove more feasible to adopt for an elastic appendage a finite-element model such as that described in the Introduction. The equations of small vibratory deformation can then be recorded from [1]† (see equation 64), and the appropriate coordinate transformation can be adopted from [1] (see equation 74). The vector $\mathbf{c}^k$ in equation (8) is then replaced by $\mathbf{c}$ as recorded in [1], equation (43), with $\mathbf{e} \equiv 0$ and summation extending only over the kth substructure. Although the dyadic $\mathbf{J}^k$ in equation (7) and the integral $(i_d/\mathrm{d}t)\int_{\mathscr{a}_k} \mathbf{p}\times\mathring{\mathbf{p}}\,\mathrm{d}m$ are not provided in [1], which treats the distributed mass finite element model with nodal bodies, they are available in [5] for the special case with all mass concentrated in the nodal bodies; see [5], equation (126) for $\mathbf{J}^k$ and equation (127) for $\mathring{\mathbf{J}}^k$, and see equation (112) or equation (114) for $(i_d/\mathrm{d}t)\int_{\mathscr{a}_k} \mathbf{p}\times\mathring{\mathbf{p}}\,\mathrm{d}m$, recognizing in each case that the superscript k indicates restriction to the kth substructure, and setting $\mathbf{\Omega}^a$ and ξ as found in [5] identically to zero. In each case the indicated expressions involve deformation variables which must be subjected to coordinate transformations and truncations, as indicated in [1] and [5].

If it pleases the analyst, he can conceal the complexities of modeling the elastic appendage by simply adopting a symbolic model in terms of mode shapes and natural frequencies. Then he can rather easily record the equations of vibration (probably using Lagrange's equations with modal coordinates for generalized coordinates), and he can evaluate the deformation-dependent terms in equations (31)–(33) in terms of unspecified mode shapes and frequencies. Of course he must face the problems of adopting an appendage model and determining its modal characteristics when he wants to face any real problem.

SUMMARY

By combining equations (9), (31), (32) and (36) with equation (64) of [1], one can obtain a generic statement of a minimum-dimension set of equations of motion of a system of $n+1$ rigid bodies interconnected by n line hinges, with a set of axisymmetric rotors and a finite-element model of an elastic appendage attached to each rigid body. In order for these equations to stand alone as a complete formulation of the problem, one must substitute the auxiliary equations (33)–(35) and (37) as well as certain specified equations from [5] into equations (31) and (32), and equation (64) of [1] must be composed from the underlying equations of that reference (equations 46, 53 and 62). Moreover, the total system of equations must for practical utilization be written in matrix form and the appendage deformations subjected to modal coordinate transformations and truncations, as these procedures are described in [1] and [5]. Although one must not underestimate the labors of proceeding from the vector–dyadic equations in this paper to a generic computer program for integrating these equations, this does appear to be a feasible task. Once the program is completed, it will have sufficient generality to encompass several spacecraft simulations for which specific numerical integration computer programs have been written in the past [7–9]. Experience with these programs provides reasonable assurance that the generic program proposed here will find application as a practical tool for the simulation of complex modern spacecraft.

† Note that the vector called $\mathbf{X}$ in [1] must be interpreted for the rth appendage of the present paper as $\mathbf{X}+\boldsymbol{\rho}^r$, with $\boldsymbol{\rho}^r$ expanded as in equation (19).

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Резюме—Создали уравнения движения для моделирования космического корабля или другой сложной электромеханической системы, которые можно изменять в теоретический комплект жестких тел, навешанных на петлях по топологии "дерева", с жесткими осесимметричными роторами и нежесткими придатками, которые прикреплены к каждому жесткому телу комплекта. Эта работа, вместе с ранее опубликованной относящейся работой об уравнениях вибрации финитныхэлементов придатков, представляет полную формулировку минимального-размера, пригодного для характерного программирования и численной интеграции ЦВМ.